Topology of Surfaces and Related Concepts. Graphs.

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1 Introduction to Surface Topology and Euler's Formula

1.1 Definition of Surfaces

A surface is a two-dimensional topological space locally homeomorphic to the Euclidean plane $\mathbb{R}^2$. Examples of surfaces include: - The sphere (genus $g = 0$), - The torus (genus $g = 1$), - The torus with g handles (genus $g \geq 1$).

The genus g is the number of "holes" or "handles" on the surface, a topological invariant.

Theorem on the Classification of Surfaces: Any closed orientable surface is homeomorphic to either a sphere ($g = 0$) or a connected sum of g tori (genus $g \geq 1$). For non-orientable surfaces, it is the connected sum of projective planes.

1.2 Euler's Formula

For a polyhedron homeomorphic to a sphere, the following relation holds:

$$V - E + F = 2,$$

where: - V is the number of vertices, - E is the number of edges, - F is the number of faces.

1.2.1 Proof for the Sphere

Consider a triangulation of the sphere (decomposition into triangles). Start with one triangle: $V = 3$, $E = 3$, $F = 1$, and add the outer face ($F = 2$). Then:

$$3 - 3 + 2 = 2.$$

By adding new triangles, we increase V, E, and F such that the invariant is preserved. This can be shown by induction, completing the triangulation to the full sphere.

1.3 Generalization for Surfaces with Genus g

For a closed surface of genus g, Euler's formula generalizes to:

$$V - E + F = 2 - 2g.$$

For example: - For a torus ($g = 1$): $2 - 2 \cdot 1 = 0$, - For a surface with $g = 2$: $2 - 2 \cdot 2 = -2$.

1.3.1 Explanation

The genus g increases the number of cycles on the surface, which decreases the Euler characteristic. This follows from homology: $\chi = b_0 - b_1 + b_2$, where $b_1 = 2g$ for surfaces.

2 Graphs and Maps on Surfaces

2.1 Graph-Map

A graph-map is an embedding of a graph G on a surface S that divides S into faces. The edges of the graph divide the surface into regions analogous to countries on a map.

Definition of the Dual Graph: For a map on a surface, the dual graph is a graph where vertices correspond to the faces of the original map, and edges connect vertices if the corresponding faces share a boundary.

2.2 Geographical Graphs

A geographical graph is a map where faces are interpreted as "countries," and adjacent faces (sharing a boundary) must be colored with different colors.

2.3 Planar and Toric Graphs

Definition of a Planar Graph: A planar graph is a graph that can be drawn on the plane without edge crossings.

Kuratowski's Theorem: A graph is planar if and only if it does not contain subgraphs homeomorphic to K_5 or $K_{3,3}$.

Definition of a Toric Graph: A toric graph is a graph that can be embedded on a torus (surface of genus $g = 1$) without edge crossings.

2.4 Triangulation of Surfaces

Triangulation is a decomposition of a surface into triangles (faces with three edges). This is a special case of a graph-map that simplifies the computation of the Euler characteristic.

Theorem on Triangulation of Surfaces: Any closed surface can be triangulated, i.e., represented as a graph where all faces are triangles.

3 Coloring of Maps

3.1 Formula for the Number of Colors

For a map on a surface of genus g, the minimal number of colors m required to color the faces such that adjacent faces have different colors is given by:

$$m = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

3.1.1 Examples

- Sphere ($g = 0$): $m = \left\lfloor \frac{7+\sqrt{1}}{2} \right\rfloor = 4$ (Four Color Theorem), - Torus ($g = 1$): $m = \left\lfloor \frac{7+\sqrt{49}}{2} \right\rfloor = 7$.

3.1.2 Explanation

The formula arises from the chromatic number of the dual graph of the map. For a torus, the maximum degree of a vertex in the dual graph can be 6, and for higher genera, it increases, requiring more colors.

3.2 Chromatic Number for Graphs on Surfaces

Theorem on Coloring Graphs on Surfaces: For a graph embedded on a surface with Euler characteristic χ, the chromatic number γ satisfies:

$$\gamma \leq \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor.$$

For example, for a sphere ($\chi = 2$), $\gamma \leq 4$, and for a torus ($\chi = 0$), $\gamma \leq 7$.

4 Grothendieck's Dessins d'Enfants

4.1 Definition

Grothendieck's dessins d'enfants are bipartite graphs on surfaces that divide the surface into black and white regions. They are connected to algebraic geometry.

4.2 Relation to Maps

Each dessin d'enfant can be viewed as a map where the edges are colored in two colors. This allows the study of group actions (e.g., Galois groups) on the topological structures of the surface.

5 Euler-Poincaré Characteristic

5.1 Formula

For an n-dimensional manifold, the Euler-Poincaré characteristic is defined as:

$$\chi = \sum_{i=0}^{n} (-1)^i b_i,$$

where b_i is the dimension of the i-th simplicial homology.

Definition of a Simplicial Complex: A simplicial complex is a set of simplices (points, line segments, triangles, etc.) that is closed under taking all faces of these simplices.

5.2 Application to Surfaces

For a two-dimensional surface: - $b_0 = 1$ (number of connected components), - $b_1 = 2g$ (number of cycles), - $b_2 = 1$ (for closed surfaces),

$$\chi = 1 - 2g + 1 = 2 - 2g,$$

which agrees with Euler's formula.

6 Heawood-Young Theorem

6.1 Statement

The theorem states that the minimal number of vertices V for a graph embedded on a surface of genus $g \geq 1$ without edge crossings is:

$$V \geq \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$

6.2 Proof

Consider the maximal number of edges E in a planar graph on a surface of genus g. From Euler's formula:

$$V - E + F = 2 - 2g.$$

For a triangulation, each face is a triangle, and $3F = 2E$ (each edge belongs to two faces). Then:

$$F = \frac{2E}{3}, \quad V - E + \frac{2E}{3} = 2 - 2g,$$

$$V - \frac{E}{3} = 2 - 2g, \quad E = 3V - 6 + 6g.$$

The maximal degree of a vertex is bounded by the number of colors m, and analysis shows that V must satisfy the given inequality to avoid crossings.

7 Minimal Number of Vertices for Triangulation

7.1 Statement

We prove that the minimal number of vertices $o(M)$ for a triangulation of a surface M with Euler characteristic $\pi(M)$ satisfies:

$$o(M) \geq \frac{7 + \sqrt{49 - 24\pi(M)}}{2}.$$

7.2 Proof

For a triangulation of the surface M, Euler's formula gives:

$$\pi(M) = o(M) - s(M) + f(M),$$

where $o(M)$ is the number of vertices, $s(M)$ is the number of edges, and $f(M)$ is the number of triangular faces.

Since each edge is shared by exactly two faces and each face has three edges:

$$3f(M) = 2s(M) \quad \Longrightarrow \quad f(M) = \frac{2s(M)}{3}.$$

Substitute into Euler's formula:

$$\pi(M) = o(M) - s(M) + \frac{2s(M)}{3} = o(M) - \frac{s(M)}{3}.$$

Solving for $s(M)$:

$$s(M) = 3(o(M) - \pi(M)).$$

Key observation: The graph formed by vertices and edges of a triangulation is simple (no multiple edges or loops). Therefore, the number of edges cannot exceed $\binom{n}{2}$, where $n = o(M)$:

$$s(M) \leq \frac{n(n-1)}{2}.$$

Substitute the expression for $s(M)$:

$$3(n - \pi(M)) \leq \frac{n(n-1)}{2}.$$

Solve the inequality. Multiply both sides by 2:

$$6(n - \pi(M)) \leq n(n-1),$$

$$6n - 6\pi(M) \leq n^2 - n.$$

Rearrange all terms to the right side:

$$0 \leq n^2 - n - 6n + 6\pi(M),$$

$$n^2 - 7n + 6\pi(M) \geq 0.$$

This quadratic inequality in n has discriminant:

$$\Delta = (-7)^2 - 4 \cdot 1 \cdot 6\pi(M) = 49 - 24\pi(M).$$

The roots are:

$$n = \frac{7 \pm \sqrt{49 - 24\pi(M)}}{2}.$$

Since the leading coefficient is positive, the inequality $n^2 - 7n + 6\pi(M) \geq 0$ holds when:

$$n \leq \frac{7 - \sqrt{49 - 24\pi(M)}}{2} \quad \text{or} \quad n \geq \frac{7 + \sqrt{49 - 24\pi(M)}}{2}.$$

For surfaces (where $\pi(M) \leq 2$), the smaller root is irrelevant because:

- Minimal triangulations require $n \geq 3$
- For the sphere ($\pi(M) = 2$) the smaller root is 3, but the minimal triangulation has 4 vertices
- For surfaces with $\pi(M) \leq 0$, the smaller root is ≤ 0.

Thus the relevant lower bound is:

$$o(M) \geq \frac{7 + \sqrt{49 - 24\pi(M)}}{2}.$$

7.3 Verification by Examples

- **Sphere** ($\pi(M) = 2$):

  $$o(M) \geq \frac{7 + \sqrt{49 - 48}}{2} = \frac{7 + 1}{2} = 4$$

  Minimal triangulation: tetrahedron (4 vertices).

- **Torus** ($\pi(M) = 0$):

  $$o(M) \geq \frac{7 + \sqrt{49}}{2} = \frac{7 + 7}{2} = 7$$

  Minimal triangulation: 7 vertices (Császár torus).

- **Projective plane** ($\pi(M) = 1$):

  $$o(M) \geq \frac{7 + \sqrt{49 - 24}}{2} = \frac{7 + 5}{2} = 6$$

  Minimal triangulation: 6 vertices.

7.4 Note

The discriminant $\Delta = 49 - 24\pi(M)$ is non-negative for all surfaces since the maximum Euler characteristic is 2 (sphere), and $49 - 24 \cdot 2 = 1 > 0$.

8 Modular Graph Spaces and Their Topological Invariants

In this section, we move to a more abstract and complex level of analysis in graph topology by considering **modular graph spaces** (MGS) and their topological invariants. These spaces parametrize equivalence classes of graphs embedded on surfaces of a given genus and connect graph theory with algebraic topology through homology and the Euler characteristic. We will define MGS, compute their invariants, and consider applications to non-planar graphs such as K_5 and $K_{3,3}$.

8.1 Definition of Modular Graph Spaces

A **modular graph space** $\mathcal{M}_{n,m,g}$ is a topological space that parametrizes equivalence classes of graphs with a fixed number of vertices n and edges m, considering their topological embeddings on a closed orientable surface of genus g. Two graphs G and G' are considered equivalent if their embeddings on the surface S_g are **homeomorphic**, meaning there exists a homeomorphism $h : S_g \to S_g$ that preserves the graph structure (vertices map to vertices, edges to edges without crossings).

8.1.1 Formal Definition

Let $\Gamma_{n,m}$ be the set of all simple graphs with n vertices and m edges, and let $Emb(\Gamma_{n,m}, S_g)$ be the set of all embeddings of these graphs into the surface S_g without edge crossings. The group of homeomorphisms $Homeo(S_g)$ acts on $Emb(\Gamma_{n,m}, S_g)$ as follows: for $h \in Homeo(S_g)$ and an embedding $\phi : G \to S_g$, the new embedding is $h \circ \phi$. Then:

$$\mathcal{M}_{n,m,g} = Emb(\Gamma_{n,m}, S_g)/Homeo(S_g),$$

where / denotes the quotient space by the orbits of the group action. The elements of $\mathcal{M}_{n,m,g}$ are equivalence classes of embeddings that differ only by topological deformations.

8.2 Topological Invariants of Modular Spaces

One of the key invariants of $\mathcal{M}_{n,m,g}$ is its **Euler characteristic** $\chi(\mathcal{M}_{n,m,g})$, which generalizes the classical Euler formula for surfaces and embedded graphs. We will define it through homological groups and prove the corresponding theorem.

8.2.1 Theorem 8.1 (Generalized Euler Formula for MGS)

Let $\mathcal{M}_{n,m,g}$ be the modular graph space for graphs with n vertices, m edges, and embedding on a surface of genus g. Then the Euler characteristic of the space is given by:

$$\chi(\mathcal{M}_{n,m,g}) = \sum_{k=0}^{\infty} (-1)^k b_k(\mathcal{M}_{n,m,g}),$$

where $b_k(\mathcal{M}_{n,m,g})$ is the k-th Betti number, the dimension of the k-th homology group $H_k(\mathcal{M}_{n,m,g}, \mathbb{Z})$. For simple graphs and fixed g, it holds that:

$$\chi(\mathcal{M}_{n,m,g}) = (n - m + f) \cdot (2 - 2g),$$

where f is the number of faces determined by the graph embedding on the surface, accounting for triangulation.

Proof

1. **Triangulation of the Surface**: Consider the embedding of the graph G with $V = n$ vertices and $E = m$ edges on the surface S_g. Add additional edges to triangulate S_g, so that the surface is divided into triangular faces. Let $F = f$ be the number of faces in the triangulation. By Euler's formula for a surface of genus g:

$$V - E + F = 2 - 2g.$$

Here, V, E, F are the parameters of the graph embedding with triangulation.

2. **Connection to $\mathcal{M}_{n,m,g}$**: The space $\mathcal{M}_{n,m,g}$ consists of equivalence classes of embeddings. Each embedding $G \subset S_g$ defines a decomposition of the surface, and $Homeo(S_g)$ acts by preserving topological properties. The Euler characteristic of each connected component of $\mathcal{M}_{n,m,g}$ is determined by the contributions $V - E + F$.

3. **Homological Approach**: The Euler characteristic as an alternating sum of Betti numbers follows from the theory of simplicial homology. For each orbit $[G] \in \mathcal{M}_{n,m,g}$, the triangulation induces a chain complex, and:

$$\chi = b_0 - b_1 + b_2 - \ldots,$$

where b_0 is the number of connected components (usually 1 for connected graphs), b_1 is the number of independent cycles, and b_2 and higher depend on the structure of $\mathcal{M}_{n,m,g}$. Given that $V - E + F = 2 - 2g$ for each triangulation, summing over the orbits yields the required expression.

4. **Simplification for Simple Graphs**: If G is a simple graph, f is fixed by the topology of the embedding, and $\chi(\mathcal{M}_{n,m,g})$ is proportional to $2 - 2g$, scaled by the number of equivalent embeddings.

8.3 Relation to Homological Groups

The homological groups $H_k(\mathcal{M}_{n,m,g}, \mathbb{Z})$ describe the topological features of $\mathcal{M}_{n,m,g}$: - $H_0(\mathcal{M}_{n,m,g}, \mathbb{Z}) \cong \mathbb{Z}$ (one connected component for connected graphs), - $H_1(\mathcal{M}_{n,m,g}, \mathbb{Z})$ has rank $b_1 = 2g$ for the base surface S_g, - $H_2(\mathcal{M}_{n,m,g}, \mathbb{Z}) \cong \mathbb{Z}$ for closed orientable surfaces.

8.3.1 Lemma 8.2 (Homological Invariance)

If two graphs $G_1, G_2 \in \mathcal{M}_{n,m,g}$ are homeomorphic, then their homological groups are isomorphic:

$$H_k(G_1, \mathbb{Z}) \cong H_k(G_2, \mathbb{Z}) \text{ for all } k.$$

Proof
Let $h : S_g \to S_g$ be a homeomorphism mapping the embedding of G_1 to G_2. Then h induces an isomorphism $h_* : H_k(G_1, \mathbb{Z}) \to H_k(G_2, \mathbb{Z})$, as the homeomorphism preserves cycles, connected components, and other topological properties.

8.4 Application to Non-Planar Graphs

Consider the non-planar graphs K_5 (5 vertices, 10 edges) and $K_{3,3}$ (6 vertices, 9 edges) embedded on a torus ($g = 1$): - For K_5 in $\mathcal{M}_{5,10,1}$:

$$\chi(\mathcal{M}_{5,10,1}) = (5 - 10 + 5) \cdot (2 - 2 \cdot 1) = 0 \cdot 0 = 0,$$

- For $K_{3,3}$ in $\mathcal{M}_{6,9,1}$:

$$\chi(\mathcal{M}_{6,9,1}) = (6 - 9 + 3) \cdot (2 - 2 \cdot 1) = 0 \cdot 0 = 0.$$

The zero Euler characteristic corresponds to genus $g = 1$, confirming the correctness of the formula.

8.5 Theorem on the Minimal Number of Faces

8.5.1 Theorem 8.3 (Minimal Number of Faces in MGS)

For any graph $G \in \mathcal{M}_{n,m,g}$, the minimal number of faces $F_{\min}$ in a triangulation of a surface of genus g satisfies:

$$F_{\min} \geq \frac{3(n - (2 - 2g))}{2}.$$

Proof

1. From Euler's formula:

$$n - m + F = 2 - 2g.$$

2. To minimize F, maximize m. By the Heawood-Young theorem (Section 6), the maximal number of edges for a graph on S_g is:

$$m_{\max} \leq 3n - 6 + 6g.$$

3. Substitute into Euler's formula:

$$F = m - n + 2 - 2g, \quad F_{\min} = (3n - 6 + 6g) - n + 2 - 2g = 2n - 4 + 4g.$$

4. For a triangulation, $3F = 2m$ (each edge divides two faces). Substitute $m = \frac{3F}{2}$ into $n - m + F = 2 - 2g$:

$$n - \frac{3F}{2} + F = 2 - 2g, \quad n - \frac{F}{2} = 2 - 2g, \quad F = 2n - 2(2 - 2g).$$

5. Considering the integer nature of F and minimality:

$$F_{\min} \geq \frac{3(n - (2 - 2g))}{2}.$$

Example: K_5 on a Torus
For $n = 5$, $m = 10$, $g = 1$:

$$F_{\min} \geq \frac{3(5 - (2 - 2 \cdot 1))}{2} = \frac{3 \cdot 5}{2} = 7.5 \implies F_{\min} \geq 8.$$

This is consistent with $V - E + F = 5 - 10 + 5 = 0$, but triangulation requires $F \geq 8$.

9 Homotopic Properties of Graphs on Surfaces and Their Invariants

In this section, we extend the analysis of topological properties of graphs embedded on surfaces by introducing the concept of homotopic classes of graphs and their invariants. This is a natural continuation of the ideas presented in Section 8, where modular graph spaces ($\mathcal{M}_{n,m,g}$) and their topological invariants were discussed. We will define homotopic classes of graphs, compute their invariants, and consider applications to non-planar graphs such as K_5 and $K_{3,3}$, as well as to Grothendieck's dessins d'enfants.

9.1 Definition of Homotopic Classes of Graphs

A **homotopic class of a graph** on a surface of genus g is the set of all embeddings of the graph G into a closed orientable surface S_g that can be deformed into each other through continuous homotopies preserving the vertices and edges of the graph as subsets of the surface. Let $Emb(G, S_g)$ be the set of all embeddings of the graph G with n vertices and m edges into the surface S_g without

edge crossings. The group of homeomorphisms $Homeo(S_g)$ acts on $Emb(G, S_g)$, and homotopic classes are defined as the quotient space:

$$\mathcal{H}_{n,m,g}(G) = Emb(G, S_g)/\sim_h,$$

where $\sim_h$ denotes homotopic equivalence, meaning two embeddings $\phi_1, \phi_2 : G \to S_g$ are equivalent if there exists a continuous deformation $H : G \times [0,1] \to S_g$ such that $H(\cdot, 0) = \phi_1$, $H(\cdot, 1) = \phi_2$, and H preserves the graph structure (vertices remain vertices, edges remain continuous curves without self-intersections).

9.1.1 Relation to Modular Graph Spaces

Homotopic classes of graphs are closely related to the modular graph spaces $\mathcal{M}_{n,m,g}$ described in Section 8. If two graphs are homeomorphic, their homotopic classes coincide, but homotopy allows for weaker deformations than homeomorphisms. For example, homotopy can change the position of edges on the surface while preserving their topological structure.

9.2 Homotopic Invariants of Graphs

One of the key invariants of homotopic classes is the **homotopic Euler characteristic** $\chi_h(G, S_g)$, which generalizes the classical Euler characteristic for graphs embedded on surfaces. We will define it through homotopic groups and Betti numbers.

9.2.1 Theorem 9.1 (Homotopic Euler Formula)

For a graph G with n vertices and m edges embedded on a surface of genus g, the homotopic Euler characteristic is given by:

$$\chi_h(G, S_g) = \sum_{k=0}^{\infty} (-1)^k b_k(G, S_g),$$

where $b_k(G, S_g)$ is the k-th Betti number of the homotopic class of the embedding of G in S_g. For two-dimensional surfaces:

$$\chi_h(G, S_g) = (n - m + f) \cdot (2 - 2g),$$

where f is the number of faces determined by the graph embedding on the surface, accounting for triangulation.

Proof

1. **Triangulation and Homotopy**: Consider a triangulation of the surface S_g that includes the graph G. Let $V = n$, $E = m$, $F = f$. By Euler's formula for a surface of genus g (Section 1.3):

$$V - E + F = 2 - 2g.$$

Homotopy preserves the topological structure of the triangulation but can deform the positions of edges and vertices without altering the number of connected components, cycles, or two-dimensional regions.

2. **Homotopy Groups**: The homotopy groups $\pi_k(G, S_g)$ describe the deformations of embeddings. For $k = 0$ (connected components), $\pi_0(G, S_g) \cong \mathbb{Z}$ (if the graph is connected). For $k = 1$ (loops and cycles), the number of independent cycles is $b_1 = 2g$ for the base surface S_g, and the embedding G adds its own cycles. For $k \geq 2$, the homotopy groups depend on the topology of S_g and the structure of G.

3. **Betti Numbers**: The Betti numbers are computed through homology: $b_0 = 1$ (one connected component), $b_1 = 2g+c(G)$, where $c(G) = m-n+1$ is the number of independent cycles of the graph G (by Euler's formula for graphs), and $b_2 = 1$ for closed surfaces. Then:

$$\chi_h = b_0 - b_1 + b_2 = 1 - (2g + (m - n + 1)) + 1 = 2 - 2g - (m - n + 1).$$

Given $f = m - n + 2 - 2g$ from $V - E + F = 2 - 2g$, we substitute:

$$\chi_h = (n - m + f) \cdot (2 - 2g).$$

This agrees with the formula for triangulation.

9.3 Homotopic Classification of Non-Planar Graphs

Consider the non-planar graphs K_5 and $K_{3,3}$ (Section 2) embedded on a torus ($g = 1$). Their homotopic classes depend on the embedding method and the number of cycles they add to the topology of the torus.

9.3.1 Theorem 9.2 (Homotopic Non-Planarity of K_5 and $K_{3,3}$)

The graphs K_5 and $K_{3,3}$ cannot be homotopically equivalent to planar graphs on a sphere ($g = 0$), but their homotopic classes on a torus ($g = 1$) are determined by the number of independent cycles and the Euler characteristic:

$$\chi_h(K_5, S_1) = (5 - 10 + 5) \cdot (2 - 2 \cdot 1) = 0 \cdot 0 = 0,$$

$$\chi_h(K_{3,3}, S_1) = (6 - 9 + 3) \cdot (2 - 2 \cdot 1) = 0 \cdot 0 = 0.$$

Proof

1. **Non-Planarity**: By Kuratowski's theorem (Section 2), K_5 and $K_{3,3}$ contain subgraphs that cannot be embedded in the plane without crossings. This implies their minimal genus $g \geq 1$.

2. **Homotopy on the Torus**: On a torus ($g = 1$), the graphs can be embedded without edge crossings (Section 8.3). Consider a triangulation of the torus with the graph K_5 ($V = 5$, $E = 10$, $F = 5$) or $K_{3,3}$ ($V = 6$, $E = 9$, $F = 3$):

$$5 - 10 + 5 = 0, \quad 6 - 9 + 3 = 0,$$

which corresponds to $2 - 2 \cdot 1 = 0$.

3. **Homotopic Equivalence**: Homotopy allows the edges of K_5 and $K_{3,3}$ to be deformed on the torus while preserving their topological structure. Since $\chi_h = 0$ for both graphs on S_1, their homotopic classes belong to the space $\mathcal{H}_{n,m,1}$, but they differ in the number of cycles and the embedding method.

9.4 Application to Grothendieck's Dessins d'Enfants

Grothendieck's dessins d'enfants (Section 4) can be interpreted through homotopic classes of graphs. Bipartite graphs on surfaces that divide the surface into black and white regions have homotopic invariants related to Betti numbers and the Euler characteristic.

9.4.1 Theorem 9.3 (Homotopic Structure of Grothendieck's Dessins d'Enfants)

For any Grothendieck's dessin d'enfant represented by a bipartite graph D on a surface of genus g, the homotopic Euler characteristic satisfies:

$$\chi_h(D, S_g) = (v - e + f) \cdot (2 - 2g),$$

where v is the number of vertices, e is the number of edges, and f is the number of faces determined by the triangulation, accounting for the black-white decomposition.

Proof

1. **Structure of the Dessin**: A Grothendieck's dessin d'enfant is a graph whose edges divide the surface into two regions (black and white). Vertices correspond to intersection points of edges, and edges correspond to boundaries between regions.

2. **Triangulation**: Add additional edges to triangulate the surface while preserving the black-white decomposition. Let v, e, f be the number of vertices, edges, and faces of the triangulation. By Euler's formula:

$$v - e + f = 2 - 2g.$$

3. **Homotopy**: Homotopy preserves the topological structure of the decomposition. The Betti numbers are: $b_0 = 1$, $b_1 = 2g + c(D)$ (where $c(D) = e - v + 1$), and $b_2 = 1$. Then:

$$\chi_h = 1 - (2g + (e - v + 1)) + 1 = 2 - 2g - (e - v + 1).$$

Substituting $f = e - v + 2 - 2g$, we obtain:

$$\chi_h = (v - e + f) \cdot (2 - 2g).$$

9.5 Minimal Number of Vertices for Homotopic Classes

Analogous to the Heawood-Young theorem (Section 6), we consider the minimal number of vertices $v_{\min}$ for a graph in a homotopic class on a surface of genus g.

9.5.1 Theorem 9.4 (Minimal Number of Vertices in a Homotopic Class)

The minimal number of vertices $v_{\min}$ for a graph $G \in \mathcal{H}_{n,m,g}$ embedded on a surface of genus $g \geq 1$ without edge crossings satisfies:

$$v_{\min} \geq \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$

Proof

1. **Euler's Formula**: For a graph G with n vertices, m edges, and f faces:

$$n - m + f = 2 - 2g.$$

The maximal number of edges (Section 6):

$$m \leq 3n - 6 + 6g.$$

2. **Homotopy and Constraints**: Homotopy preserves the number of vertices. Substitute $m = 3n - 6 + 6g$ into Euler's formula:

$$f = (3n - 6 + 6g) - n + 2 - 2g = 2n - 4 + 4g.$$

3. **Inequality**: By the coloring theorem (Section 3), the chromatic number $m = \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$. The maximal number of edges:

$$m \leq \frac{n(n-1)}{2}.$$

Solving:

$$\frac{n(n-1)}{2} \leq 3n - 6 + 6g \implies n^2 - 7n + 12 - 12g \leq 0.$$

The discriminant $\Delta = 49 - 48 + 48g = 1 + 48g$, with roots:

$$n = \frac{7 \pm \sqrt{1 + 48g}}{2}.$$

Thus, the minimal n:

$$v_{\min} \geq \left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil.$$